Maximin share allocation for two types of items

Yu Chen, Kai Jie Wong and Xiaorui Zhang

National University of Singapore

Setting

Setting for **chores**

- Set of n agents N
- Indivisible chores $M = \{c_1, \dots, c_m\}$
- Agent i has disutility $d_i(c) > 0$ for chore c.
- Additivity: $d_i(S) = \sum_{c \in S} d_i(c)$ for all $S \subseteq M$.

Fairness notions

- Proportionality: Each agent gets at most $1/n$ of total disutility
- Maximin share (MMS): "If I divide the chores into n bundles and choose last, what is the best bundle I can guarantee myself?"

$$\mathrm{MMS}_i := \min_{(X_1,\dots,X_n) \in \Pi} \max_{j \in [n]} d_i(X_j)$$

- An allocation is MMS if $d_i(X_i) \leq \mathrm{MMS}_i$ for all agents i.
- **Does not always exist when** $n \geq 3$

Two-chore-type instances

- Chore set $M = A \cup B$ (“Type A” vs “Type B”)
- $c \in A \implies d_i(c) = \alpha_i \in \mathbb{R}_{>0}$
- $c \in B \implies d_i(c) = \beta_i \in \mathbb{R}_{>0}$

	c_1	c_2	c_3	c_4	c_5
agent 1	α_1	α_1	α_1	β_1	β_1
agent 2	α_2	α_2	α_2	β_2	β_2
agent 3	α_3	α_3	α_3	β_3	β_3

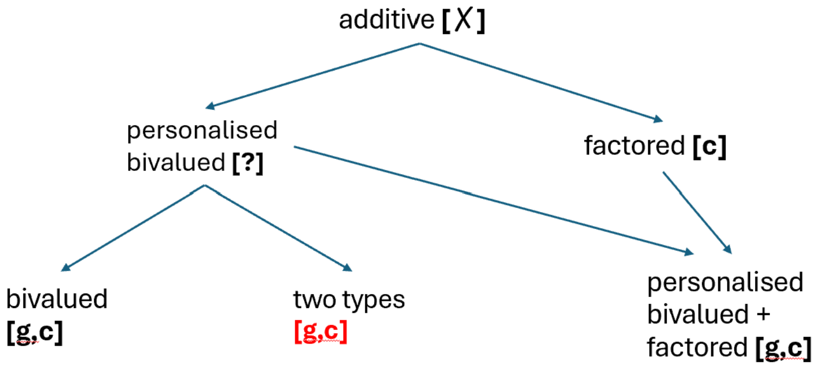
Any two-chore-type fair division instance can be succinctly defined as

$$\mathcal{I} = (a, b, N)$$

where $a := |A|$, $b := |B|$, and N is the agent set.

Contributions

[✗]: does not exist, [g]: exists for goods, [c]: exists for chores, [?]: open



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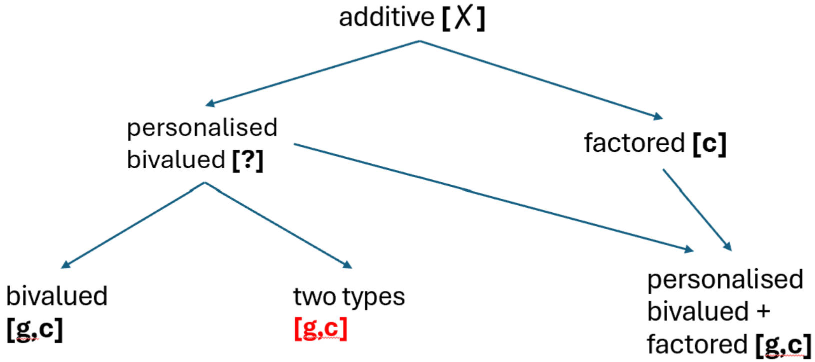


Using a **geometric perspective** from **job scheduling**, we show existence + polytime algorithm for MMS when there are two types of chores (or goods/mixed).

Connection to job scheduling

Recall that

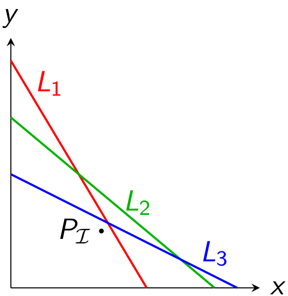
$$\mathrm{MMS}_i := \min_{(X_i)_{i\in N}} \max_{j\in N} d_i(X_j)$$

- Job scheduling: Given set of processors + jobs with processing times, find the schedule with the minimum makespan.
- Identical valuations $\implies$ MMS allocation problem = makespan minimization
- **Polynomial time if there are only two job lengths** (McCormick et al. [SODA'97])

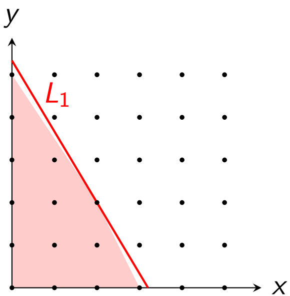
Geometric setting

Define L_i to be the line $\alpha_i x + \beta_i y = \text{MMS}_i$. Consider feasible fractional bundles

$$R_i := \{(x, y) \in \mathbb{R}^2_{\geq 0} : \alpha_i x + \beta_i y \leq \text{MMS}_i\}.$$



Instance with $n = 3$



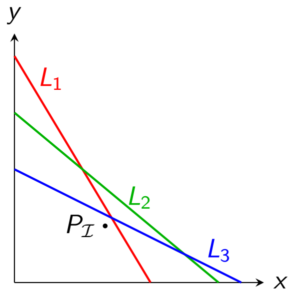
Shaded is $IH_1 = \text{conv}(R_1 \cap \mathbb{Z}^2)$

"Feasibility point": $P_{\mathcal{I}} := (\frac{a}{n}, \frac{b}{n})$.
Integer hull: $IH_i := \text{conv}(R_i \cap \mathbb{Z}^2)$

Geometric setting

Observation 1: Any integral allocation $\{(x_1, y_1), \dots, (x_n, y_n)\}$ must correspond to a polygon with centroid $P_{\mathcal{I}} = (\frac{a}{n}, \frac{b}{n})$.

Observation 2: $P_{\mathcal{I}} \in \cap_{i \in N} IH_i$



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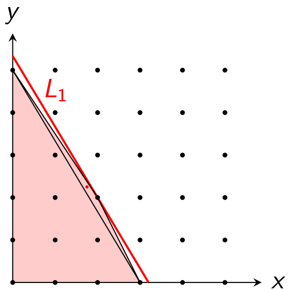


Shaded is $IH_1 = \mathrm{conv}(R_1 \cap \mathbb{Z}^2)$

New Objective: Find $(x_1, y_1) \in R_1 \cap \mathbb{Z}^2, \dots, (x_n, y_n) \in R_n \cap \mathbb{Z}^2$ with centroid $P_{\mathcal{I}}$.

Geometric setting

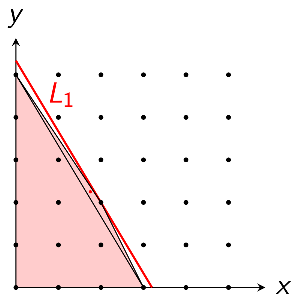
Lemma (McCormick et al., 1997)

Suppose $P_{\mathcal{I}} = (\frac{a}{n}, \frac{b}{n}) \in IH_i$. Then

- $P_{\mathcal{I}}$ lies in some *unimodular* (area 1/2) triangle $\triangle ABC$ with vertices in $R_i \cap \mathbb{Z}^2$
- There are integral multiplicities $n_1 + n_2 + n_3 = n$ such that $n_1 A + n_2 B + n_3 C = (a, b)$

and this triangle can be found in polynomial time.

$\implies$ Each agent has an MMS allocation (for themselves) which uses at most three different types of bundles



Algorithm (Sketch)

Idea: The algorithm works recursively.

- Start with $P_{\mathcal{I}} \in \cap_{i \in N} IH_i$
- Find MMS-feasible bundles (from a well-chosen agent's MMS triangle) to assign to some agents
- Show that $P_{\mathcal{I}'} \in \cap_{i \in N'} IH_i$.

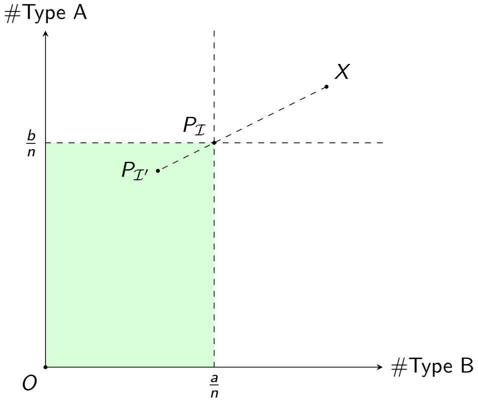
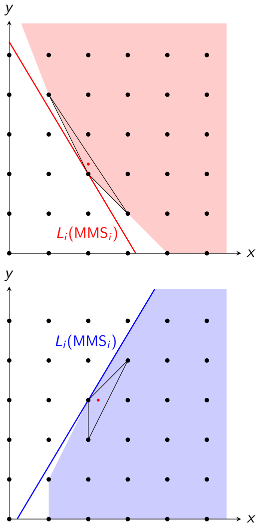


Figure: Example of easy case

Conclusion

- Extends to goods, or one objective good + one objective chore
- Does not extend easily to > 2 types: $P_{\mathcal{I}}$ may not necessarily lie within some unimodular lattice simplex under the hyperplane L_i.
- **Future work:**
  - "Two-type" valuation curves $d_i(x, y) := f(x, y)$ more general than additive?
  - Insight from more special cases in job scheduling literature?



Thank you!

Example

						MMS
Alice	20	15	10	30	40	
Bob	10	30	40	20	15	
Charlie	30	25	10	15	20	

Example

						MMS
Alice	20	15	10	30	40	40
Bob	10	30	40	20	15	40
Charlie	30	25	10	15	20	35

Example

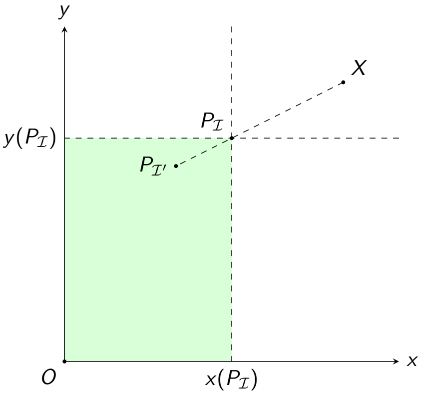
						MMS
Alice	20	15	10	30	40	40
Bob	10	30	40	20	15	40
Charlie	30	25	10	15	20	35

Proof (Case 1)

Case 1: Some agent $j \in N$ has a feasible bundle X "top-right" of $P_{\mathcal{I}}$

Assign X to agent j. For remaining instance $\mathcal{I}'$,
$P_{\mathcal{I}'} = (\frac{a-x(X)}{n-1}, \frac{b-y(X)}{n-1})$ will lie "bottom-left" of $P_{\mathcal{I}}$.

Recall $P_{\mathcal{I}} \in \cap_{i \in N} IH_i$.
Hence $P_{\mathcal{I}'} \in \cap_{i \in N \setminus \{j\}} IH_i$ (see green rectangle),
we can recurse.



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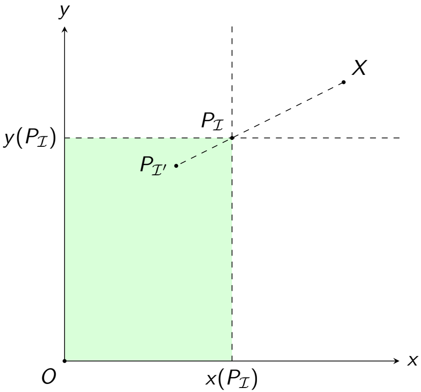


From now we may assume that no agent has a feasible bundle "top-right" of $P_{\mathcal{I}}$.

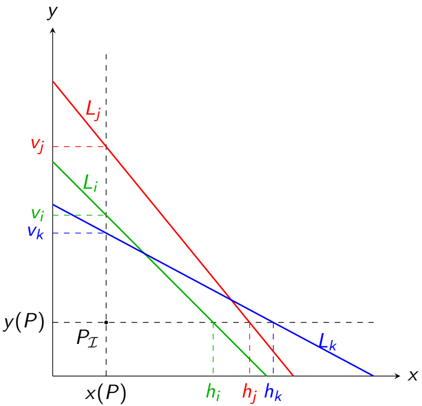
Proof

Let v_i be the value of y so that $\alpha_i \cdot x(P_{\mathcal{I}}) + \beta_i \cdot y = \text{MMS}_i$
Let h_i be the value of x so that $\alpha_i \cdot x + \beta_i \cdot y(P_{\mathcal{I}}) = \text{MMS}_i$.

Define a partial order $\preceq$ on N so that for $i, j \in N$,

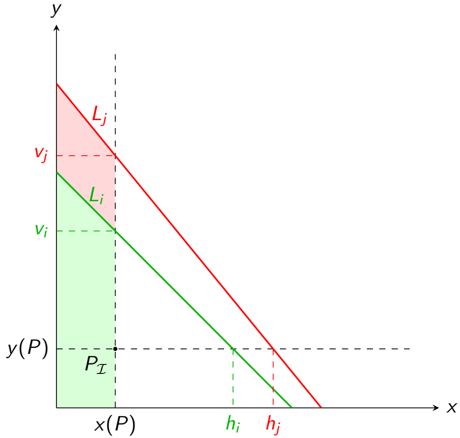
$$i \preceq j \iff v_i \leq v_j \text{ and } h_i \leq h_j.$$

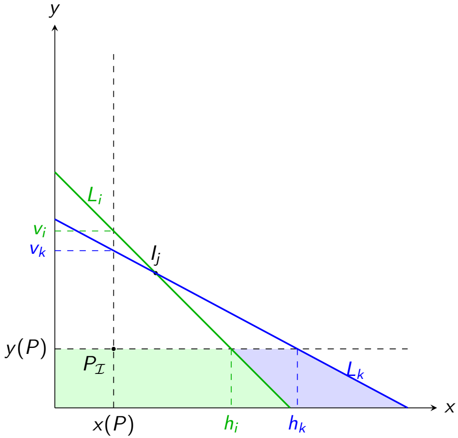
Fix $i \in N$ to be a minimal agent under $\preceq$.



Proof

Define "steep" agents $S := \{j \in N \setminus \{i\} : \frac{\beta_j}{\alpha_j} \le \frac{\beta_i}{\alpha_i}\}$, "gentle" agents $G := \{j \in N \setminus \{i\} : \frac{\beta_j}{\alpha_j} > \frac{\beta_i}{\alpha_i}\}$



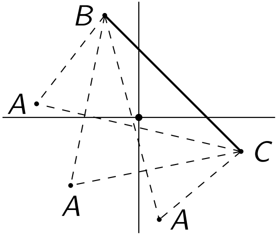


Observation: For all $j \in S$, $R_i^{\text{left}} \subseteq R_j^{\text{left}}$, and for all $k \in G$, $R_i^{\text{below}} \subseteq R_k^{\text{below}}$.

Proof (Case 2)

Observation: For all $j \in S$, $R_i^{\text{left}} \subseteq R_j^{\text{left}}$, and for all $k \in G$, $R_i^{\text{below}} \subseteq R_k^{\text{below}}$.

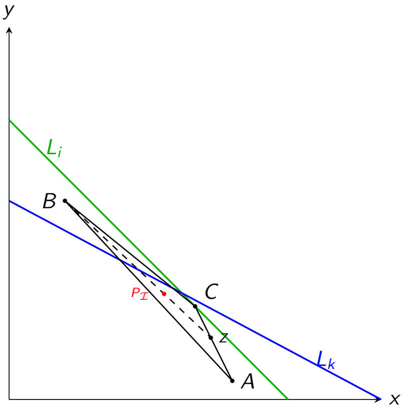
Case 2: Let $\triangle ABC$ be the MMS triangle for agent i with multiplicities $n_A + n_B + n_C = n$. WLOG let B be "top-left" of $P_{\mathcal{I}}$, C be "bottom-right" of $P_{\mathcal{I}}$, and A lies under BC.



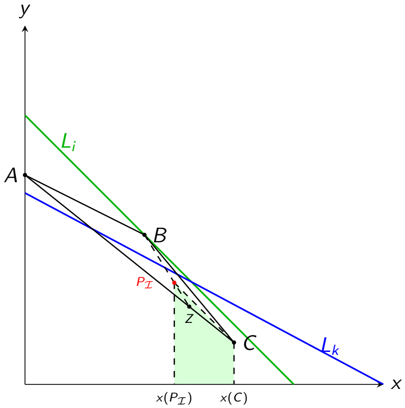
By Observation, bundles B are feasible for agents in S and bundles C are feasible for agents in G.

Proof (Case 2a)

Case 2a: $n_B \geq |S|$. Assign each agent in S the bundle B.



(a) $y(A) \leq y(P_{\mathcal{I}})$. Red point is $P_{\mathcal{I}}$. $P_{\mathcal{I}'}$ lies on $\overline{P_{\mathcal{I}}Z}$ where Z is on $\overline{AC}$.



(b) $y(A) > y(P_{\mathcal{I}})$. Red point is $P_{\mathcal{I}}$. $P_{\mathcal{I}'}$ lies within region bounded by $(x(P_{\mathcal{I}}), 0), P_{\mathcal{I}}, C, (x(C), 0)$.)

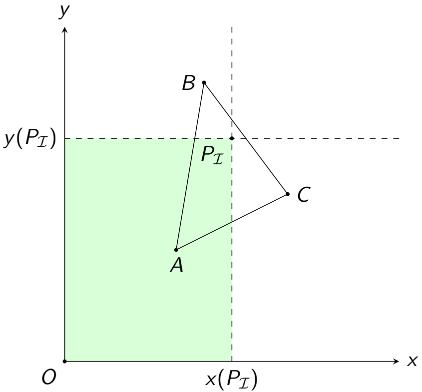
In either case, $P_{\mathcal{I}'} \in \cap_{j \in N \setminus S} IH_j$ by convexity, and we can recurse.

Proof (Case 2b)

Case 2b: $n_C \geq |G|$.
Assign each agent in G the bundle C. Proof is symmetric to Case 2a.

Proof (Case 2c)

Case 2c: $n_B < |S|, n_C < |G|$ and A lies "bottom-left" of $P_{\mathcal{I}}$.
Assign all bundles B to agents in S, all bundles C to agents in G, and all bundles A to the remaining agents.



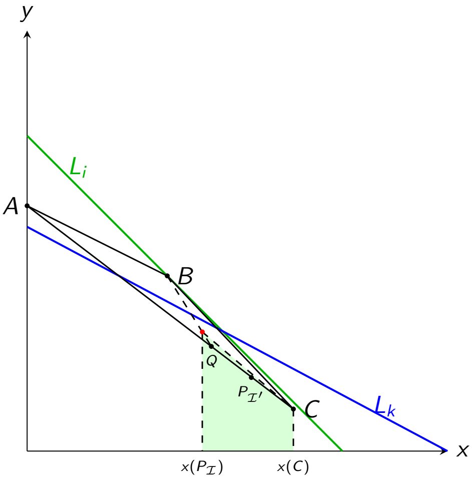
Since A is "bottom-left" of $P_{\mathcal{I}}$ it is feasible for any agent

Proof (Case 2d)

Case 2d: $n_B < |S|, n_C < |G|$ and A lies "top-left" of $P_{\mathcal{I}}$.

Assign all n_B bundles B to agents in S, and $|S| - n_B$ copies of A to remaining agents in S.

Let $Q := \frac{n_A \cdot A + n_C \cdot C}{n_A + n_C}$ be weighted centroid after removing bundles B. Q lies along $\overline{AC}$.
$P_{\mathcal{I}'}$ must lie along $\overline{QC}$, within green region.



Proof (Case 2e)

Case 2e: $n_B < |S|, n_C < |G|$ and A lies "bottom-right" of $P_{\mathcal{I}}$.
Assign all n_C bundles C to agents in G, and $|G| - n_C$ copies of A to the remaining agents in G. Proof is symmetric to Case 2d. $\square$

Hence, in all cases, we are able to assign feasible bundles to some subset of agents and recurse without increasing any agents' MMS, so an MMS allocation exists by induction. The allocation procedure can be done in polynomial time.

Thank you!